TUNING AND TEMPERAMENT: A PROBLEM OF RATIOS

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1 Introduction: A Problem of Ratios

Suppose you're tasked to solve the equation $(\frac{3}{2})^x = (\frac{2}{1})^y$ for $x, y \in \mathbb{N}$. Even upon a cursory glance, the problem is insoluble, since 3^x can't ever equal 2^{x+y} due to the unique decomposition of integers into prime factors. However, this is the problem that musicians have faced since the time of the ancient Greeks. An interval, or the distance between two notes, is defined as a ratio between the frequency of those notes. The two simplest intervals are the octave and the fifth, which are defined by the ratios $\frac{2}{1}$ and $\frac{3}{2}$, respectively. In order to use both of these intervals with notes of fixed frequency, minor adjustments must be made. This, in short, is the problem of *temperament*. Where can we hide these adjustments so that they're least noticeable? How can we approximate these ratios on physical instruments? How can we minimize this problem of ratios?

2 In the Beginning: Pythagorean Intervals

How, then, did this all get started? Why were intervals defined in this mutually incompatible way? Our story begins as many do, shrouded in myth and legend.

The ancient Pythagoreans were a mystic cult current around 500 BCE. They worshipped the whole numbers and found ratios between small numbers to be particularly simple and stable [2]. Legend has it that Pythagoras was working by a blacksmith and heard consonant ("pleasing") sounds coming from the forge. He weighed the hammers and found their weights to be in simple ratios: 4 : 3 : 2 : 1. He then corroborated this result by weighing down equal lengths of string with weights in the ratio of 4 : 3 : 2 : 1. However, both of these stories are patently false since actually performing these experiments does not yield the expected intervals . A more realistic (that is, physically plausible and reproducible) explanation suggests he may have struck "bronze discs of equal diameter but different thicknesses" in the same ratio of $4:3:2:1$ [7]. Regardless of the exact method, these ratios of simple whole numbers became the ideal for pure, or "Pythagorean" intervals.

In addition to Pythagorean mysticism, there is also an acoustic basis for favoring these ratios of simple whole numbers, due to a phenomenon known as the "harmonic series". Simply put, any pitch played on an acoustic instrument of frequency a will also produce "harmonics" of frequency $2a, 3a, 4a, \ldots$ These harmonics are present to various degrees depending on the instrument being played (this, in fact, is why we can tell the difference between a violin and a clarinet - the strength of various harmonics allows us to differentiate the two), but they are all always present to some degree [3]. We can tack on to this the phenomenon known as "beating". When two close frequencies are sounded simultaneously, they reinforce each other (that is, "beat") according to the difference in their frequencies. For instance, pitches that sound at 400 Hz (cycles per second) and 402 Hz will beat at a rate of twice per second [3]. This beating is aurally perceptible as a pulse or difference in volume and is often considered undesirable. Thus, intervals produced that are close to (but not exactly) the simple whole number ratios generated by the harmonic series will produce "beats" in the harmonics.

As an example, consider a pure interval of a fifth, in the ratio $\frac{3}{2}$. Let $x = 2a$ and $y = 3a$. Then the third harmonic of $x (= 3 * 2a = 6a)$ and the second harmonic of $y (= 2 * 3a = 6a)$ will reinforce each other. But, if y is slightly out of tune, say $y = 3.01a$, then its second harmonic becomes $6.02a$, which will beat with the third harmonic of x.

Thus, while the Pythagoreans favored these intervals of simple ratios for their elegance and mystic properties, there was also a strong acoustic bias for choosing them as well, which was likely known to practicing musicians of the time.

3 The Later Greeks: Refining the Ideal

While singers and string players (and often wind players) can often adjust the pitch of each note to make it "in tune" according to the Pythagorean ideal, some instruments are designed with fixed, immovable pitches. One such instrument popular among the ancient Greeks was the lyre, which functioned like a stripped-down version of the modern-day harp [7]. How, then, can we tune its strings to minimise the problem of ratios?

The Pythagoreans elected to tune based on the principle of using entirely perfect fifths. We already know that this will not line up with perfect octaves, and the Pythagoreans deduced this as well. After producing twelve fifths, ($(\frac{3}{2})^{12} = 129.746\ldots$), they decided it was close enough to seven octaves ($(\frac{2}{1})^{7} = 128$). This meant that the last fifth interval was unusably large (so large that it became known as the "wolf" fifth in reference to the howling of wolves!), but the other fifths remained intact [3]. The next smallest interval, in the ratio of $\frac{4}{3}$, is known as a fourth. It has exactly the same problems of the fifth, just in the opposite direction: when you increase the bottom note of a fifth by an octave you get a fourth: Instead of a ratio of $3:2$, you double the 2 to get a ratio of $4:3$. We then come to the problem of the major third, in the ratio of $5:4$. The way intervals are constructed under the Pythagorean system (the rationale of which is beyond the scope of this paper), four fifths less two octaves should yield a major third. That is, $(\frac{3}{2})^4 * (\frac{2}{1})^{-2} = 1.265625$, which differs wildly from the value $\frac{5}{4} = 1.25$. This was not a satisfactory solution.

Starting with the three most basic consonances of the octave ($\frac{2}{1}$), fifth ($\frac{3}{2}$), and fourth ($\frac{4}{3}$), it is easy to see that a stacked fifth and fourth exactly equal an octave (since $\frac{3}{2} * \frac{4}{3} = \frac{2}{1}$). What about the difference between a fifth and a fourth? This smaller interval, called a "tone" equals $\frac{3}{2} \div \frac{4}{3} = \frac{9}{8}$. It would be very neat and tidy if a tone fit exactly into the three basic consonances.

According to Ptolemy, the followers of Aristoxenus convinced themselves that the fourth was equal to two and a half tones, the fifth was equal to three and a half tones, and the octave was thus equal to six tones as follows [5]:

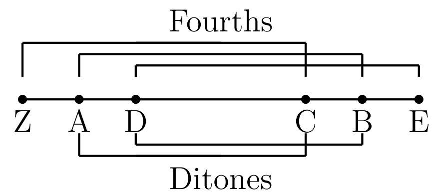

Figure 1.

Referring to Figure 1, start with a fourth between the pitches A and B, where A is lower than B. Construct a ditone (two tones stacked on top of each other $= (\frac{9}{8})^2 = \frac{81}{80}$) AC up from A and a ditone BD down from B. Then AD = CB is the difference between the ditone and the fourth. Construct a fourth DE up from D and a fourth CZ down from C. Then AD = BE (since AB and DE are both fourths, one can simply remove the common ditone BD from the middle) and likewise CB = AZ. Then BE = AD = CB = AZ. Now, the largest interval thus formed, EZ, aurally sounds like a fifth. Then the difference between EZ (a fifth) and AB (a fourth) is known to be a tone and is equal to AZ + BE. Since AZ and BE are themselves equal, each of the four intervals BE, AD, CB, and AZ must be exactly half a tone. Then AB (a fourth) consists of two and a half tones and EZ (a fifth) consists of three and a half tones. Then an octave (consisting of a stacked fifth and fourth) consists of six tones.

However, Ptolemy goes on to show that this cannot possibly be the case by means of ratios. On an eight-stringed lyre, he asks us to tune the first seven strings in six intervals of a tone (that is, a ratio of $\frac{9}{8}$) and to to tune the final string in an octave with the first (that is, a ratio of $\frac{2}{1}$). He claims that the seventh and eighth strings will be very close, but the seventh string (six tones higher than the first) will always sound slightly higher than the last string (the perfect octave). This can easily be seen by taking the exact ratio a tone is supposed to be and multiplying it by itself six times: $(\frac{9}{8})^6 = \frac{531{,}441}{262{,}144} = 2.027\cdots > 2$. Thus, Aristoxenus's supposed “proof” relied on aural perception (specifically, in the step where EZ sounds like a fifth) rather than strict mathematical rigor [5].

Following the recognition of the impossibility of equally dividing the scale - a regular series of intervals that fit within an octave - by rational means and trying to match the actual pitches used by practicing musicians, a number of competing ratio-scales were put forward by various theorists [5], but the details of these are better left to scholars of ancient Greek music theory. For now, we turn to somewhat more modern European approaches to practically dealing with the problem of ratios.

4 Zarlino: Mechanical Constructions of Geometric Means

Gradually, music began to be played in more and more keys (that is, more and more notes served as the base of theoretically identical scales). Thus, it became more and more desirable to even out the intervals so that music played in different keys would sound exactly the same. That is, even though the actual frequency of a note might change, its ratio relationship to other notes would remain identical. The sixteenth-century Italian scholar Gioseffo Zarlino describes two mechanical devices for exactly halving intervals - that is, finding geometric means [1].

In order to exactly split in an interval into two equal parts, we must find the geometric mean: that is, given frequencies a and c, we wish to find frequency b such that $\frac{a}{b} = \frac{b}{c}$. Since, all else being equal, the frequency of a vibrating string is exactly proportional to its length, we can transfer this problem over into the domain of geometry. Since Medieval times, the monochord was a popular “instrument” for measuring intervals. As one might surmise from its name, it consisted of a single string along with a movable piece (called the bridge) that divided the string into two parts, thus forming an interval [6]. The monochord would have been well-known to Zarlino. Given an interval on a monochord, Zarlino describes a method to find a length that exactly halves the interval - that is, the geometric mean [1].

The following proof is adapted from a publication from the Fourth Diderot Mathematical Forum [1]:

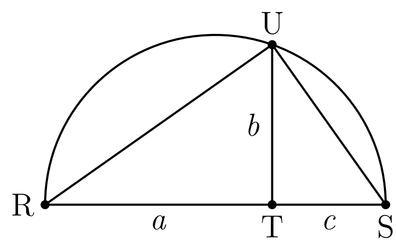

Figure 2.

Referring to Figure 2, we are given a monochord $\overline{RS}$ with the movable bridge placed at T, generating lengths a and c, as shown. Find the center of the monochord (a simple compass and straightedge construction known from the time of Euclid [2]) and construct a semicircle about the monochord. Construct a perpendicular from the point T (another compass and straightedge construction) and mark the intersection with the semicricle, U. Then $\overline{UT}$ has length b such that $\frac{a}{b} = \frac{b}{c}$.

Proof. Construct $\overline{SU}$ and $\overline{UR}$. Per Thales of Greek antiquity, any angle inscribed in a semicircle is a right angle [2]. Then, $\angle RUS$ is right. $\angle RTU$ and $\angle STU$ are right by construction. By virtue of shared angles, right triangle RTU is similar to right triangle RUS, and right triangle RUS is similar to right triangle UTS. By transitivity of similarity, triangles RTU and UTS are similar. By proportionality of similar triangles, $\frac{\overline{RT}}{\overline{TU}} = \frac{\overline{TU}}{\overline{TS}}$. That is, $\frac{a}{b} = \frac{b}{c}$. Q.E.D.

Zarlino also cites the mesolabium of Eratosthenes, a device that can be used to find cube roots. This could be used, for instance, to divide an octave into three exactly equal major thirds. (Recall that a Pythagorean major third was defined as the interval $\frac{5}{4}$; $(\frac{5}{4})^3 = \frac{125}{64} =$

1.953125, which is not quite equal to the octave - that is, $\frac{2}{1}$. The mesolabium consisted of three identical movable rectangular plates that could be slid along two parallel frames. This device can find cube roots as follows (adapted from David Burton's text *The History of Mathematics: An Introduction* [2]):

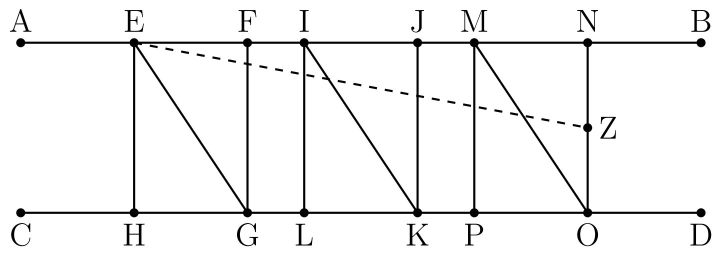

Figure 3.

Referring to Figure 3, we are given parallel frames $\overline{AB}$ and $\overline{CD}$ with movable rectangular plates EFGH, IJKL, and MNOP. We tie a string to point E and draw it taut to a point Z on $\overline{NO}$ such that $\frac{\overline{HE}}{\overline{OZ}}$ is the ratio to be divded into three equal ratios. For instance, if this ratio is $\frac{2}{1}$ then the mesolabium will find $\sqrt[3]{2}$ and $\sqrt[3]{4}$ since $\frac{2}{\sqrt[3]{4}} = \frac{\sqrt[3]{4}}{\sqrt[3]{2}} = \frac{\sqrt[3]{2}}{1}$.

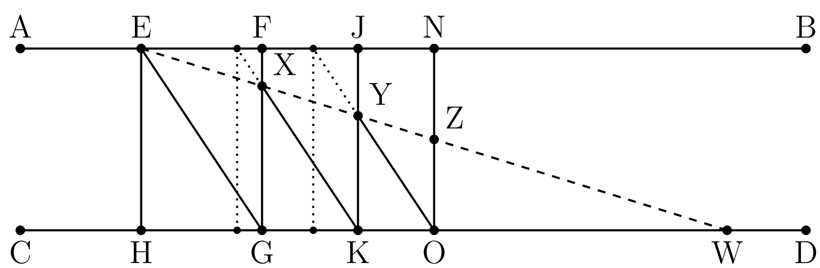

Figure 4.

Figure 4 shows the plates slid underneath each other such that the points X and Y (the intersection of a plate edge with the next plate's diagonal) fall along the taut string $\overline{EZ}$. Then $\overline{KY}$ and $\overline{GX}$ are the geometric means.

Proof. Let the string be drawn taut to the lower frame $\overline{CD}$ at point W. Since $\overline{GE}$, $\overline{KI}$, and $\overline{ML}$ are parallel as are $\overline{HE}$, $\overline{GF}$, $\overline{KJ}$, and $\overline{ON}$ (which are also perpendicular to $\overline{CD}$, we can establish many similar triangles: Triangles ZWO, YWK, XWG, and EWH are similar, as are triangles YWO, XWK, and EWG. By proportionality of similar triangles YWK and XWG, we see that $\frac{\overline{KY}}{\overline{GX}} = \frac{\overline{WK}}{\overline{WG}}$. By proportionality of similar triangles XWK and EWG, we see that $\frac{\overline{WK}}{\overline{WG}} = \frac{\overline{WX}}{\overline{WE}}$. By proportionality of similar triangles XWG and EWH, we see that $\frac{\overline{WX}}{\overline{WE}} = \frac{\overline{GX}}{\overline{HE}}$. By combining these three equalities, we yield $\frac{\overline{KY}}{\overline{GX}} = \frac{\overline{GX}}{\overline{HE}}$. By analogous reasoning, we can conclude that $\frac{\overline{OZ}}{\overline{KY}} = \frac{\overline{KY}}{\overline{GX}}$. Then $\frac{\overline{OZ}}{\overline{KY}} = \frac{\overline{KY}}{\overline{GX}} = \frac{\overline{GX}}{\overline{HE}}$. Q.E.D.

5 Schröter: Linear Algebra Yields Equal Temperament

While Zarlino was able to present constructions for precisely splitting intervals, his representation elsewhere of the twelve-note scale still relied on ratios of small whole numbers. As composers began to treat all pitches nearly identically, it became more and more desirable to make all the intervals exactly equal - a system known as *equal temperament*.

Schröter was an eighteenth century harpsichord maker and a member of the *Societät der Musikalischen Wissenschaften*, "whose very aim was precisely collaboration in the fields of music and mathematics" [1]. He presents the following algorithm [1]:

Start with a series of twelve integers. Sum them and use that number to start a new line. Add the first number of both lines to get the second number of the second line. Continue on in this manner until you have filled the second line. Now sum the second line to get the first number of the third line. Sum the first number of the second and third lines to get the second number of the third line, so on and so forth... As an example (as presented in [1]):

1	2	2	2	2	2	2	2	3	3	3	3	SUM: 27
27	28	30	32	34	36	38	40	42	45	48	51	SUM: 451
451	478	506	536	568	602	638	676	716	758	803	851	

Now, this might seem irrelevant to our goals at first, but let's take a look at what we get when we divide the second line by its first member, 27.

$$1 \quad \tfrac{28}{27} \quad \tfrac{10}{9} \quad \tfrac{32}{27} \quad \tfrac{34}{27} \quad \tfrac{4}{3} \quad \tfrac{38}{27} \quad \tfrac{40}{27} \quad \tfrac{14}{9} \quad \tfrac{5}{3} \quad \tfrac{16}{9} \quad \tfrac{17}{9} \quad (2)$$

This is already reminiscent of some of the Pythagorean intervals we dealt with earlier - $\frac{10}{9}$ approximates the $\frac{9}{8}$ tone, the perfect fourth $\frac{4}{3}$ is present, and the perfect fifth $\frac{3}{2}$ is approximated by $\frac{40}{27}$. When you divide the third line by its first member, 451, you get a series of ratios that is incredibly close to an equally tempered scale. How can this be?

Proof. Consider this as a problem of linear algebra. The first line of the table can be considered as a 1-dimensional column vector, $\vec{x}$. The process of successive adding is exactly equivalent to the multiplication $A\vec{x}$, where A is

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 2 & 1 & \dots & 1 & 1 \\ 2 & 2 & \ddots & 1 & 1 \\ \vdots & & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & 2 & 1 \end{bmatrix}$$

Note that the multiplication by the first row of A yields the sum of the first row of the table (as required), and multiplication by each subsequent row adds on each member of the first row of the table in turn.

We will now use the power method as described in [4]. Assuming that A has a dominant eigenvalue, repeated left multiplication of any $\vec{x}$ of the proper dimensions (that is, $12x1$)

by A will yield the dominant eigenvector. Then, we wish to show first that A has a dominant eigenvalue and then that the dominant eigenvector is in the proportions of an equally tempered scale - that is, where the ratio between each successive pair is equal - that is,

$$1 \quad 2^{\frac{1}{12}} \quad 2^{\frac{2}{12}} \quad 2^{\frac{3}{12}} \quad 2^{\frac{4}{12}} \quad 2^{\frac{5}{12}} \quad 2^{\frac{6}{12}} \quad 2^{\frac{7}{12}} \quad 2^{\frac{8}{12}} \quad 2^{\frac{9}{12}} \quad 2^{\frac{10}{12}} \quad 2^{\frac{11}{12}}$$

First, what is the dominant eigenvalue of A? Generally speaking, the dominant eigenvalue is the eigenvalue with the strictly largest absolute value [4]. The characteristic equation of A is $(1+\lambda)^{12} = 2\lambda^{12}$. Rewritten, we get $\lambda = \frac{1}{\sqrt[12]{2}-1}$. Considering the twelfth roots of unity, we end up with a basis of twelve eigenvectors of the form $\lambda_j = \frac{1}{\sqrt[12]{2}e^{\frac{2\pi i * j}{12}}-1}$. As seen in Figure 5, the denominator of λ_0 has the smallest absolute value, so λ_0 is the dominant eigenvalue.

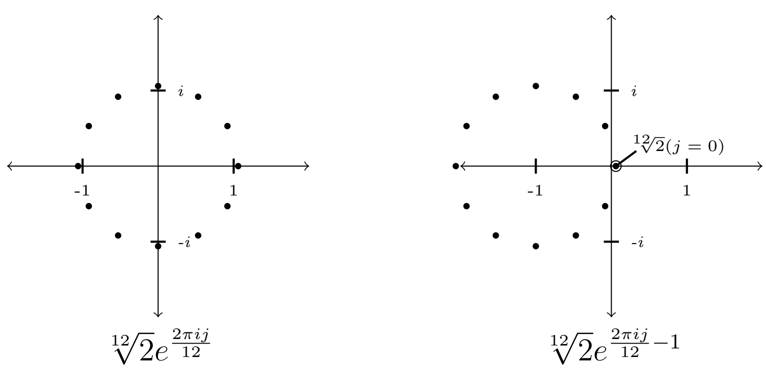

Figure 5.

We can then verify by more tedious mathematics that

$\vec{u_0} = 1 \quad 2^{\frac{1}{12}} \quad 2^{\frac{2}{12}} \quad 2^{\frac{3}{12}} \quad 2^{\frac{4}{12}} \quad 2^{\frac{5}{12}} \quad 2^{\frac{6}{12}} \quad 2^{\frac{7}{12}} \quad 2^{\frac{8}{12}} \quad 2^{\frac{9}{12}} \quad 2^{\frac{10}{12}} \quad 2^{\frac{11}{12}}$ is an eigenvector of λ_0. The power method then states that repeated left multiplication of any vector $\vec{x}$ by A will quickly approximate a multiple of $\vec{u_0}$ - that is, will produce large whole-number ratios that rapidly approximate a perfectly equally-tempered scale.

6 In Conclusion: A Modern-Day Perspective

We've examined both ancient Greek and later European methods for constructing both intervals and scales - how has this legacy come down to us today? It turns out that modern musicians most commonly use equal temperament, as composers continued to write music that treated all notes roughly equally. Indeed, virtually all modern fixed pitch instruments are tuned to equal temperament. However, as interest has recently grown in historical performance practice, various other systems of temperament espoused throughout history are finding a resurgence. Meanwhile, those who are able to adjust their pitch (wind players, string players, and vocalists) often still aspire to the purity of Pythagorean intervals when playing in ensembles. While no system will ever solve the problem of ratios that temperament presents, mathematics allows us to accurately quantify and construct the wide variety of solutions that are beginning to more peacefully coexist.

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