GEORG CANTOR:

FACING THE INFINITE

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1 Early Life and Work

Georg Cantor (1845 - 1918) was born in St. Petersburg, Russia to German parents. Although he spent his early years in Russia, the family moved back to Germany when Georg was about ten. Given these circumstances, he is regarded today as a German mathematician, and this is indeed how he viewed himself. Encouraged by his father, a devout Lutheran, Georg did well in his studies both at boarding school and trade school. However, in 1862 he sought his father's blessing to devote his life not to trade (his father was a decently successful businessman) but rather to mathematics. Upon receiving said blessing, he wrote to his father: "I hope that you will still be proud of me one day, dear Father, for my soul, my entire being lives in my calling; whatever one wants and is able to do, whatever it is toward which an unknown, secret voice calls him, *that* he will carry through to success!" [3]

Thus, Cantor began his studies at the University of Berlin, under the great German mathematicians Ernst Kummer, Leopold Kronecker, and Karl Weierstrass. His studies focused on number theory, producing a dissertation "on indeterminate equations of the second degree". He taught mathematics briefly at a girls' school, attended the Schellbach seminar for math teachers in 1868, and produced a *Habilitationsschrift* (required to become a university lecturer), again dealing with number theory. Once he secured a position at the University of Halle, however, his research turned towards analysis under the encouragement of senior faculty member Edward Heine [3].

Between 1870 and 1872, Cantor published a series of results on the unique representation of functions by trigonometric series, even if the function diverges at an infinite number of points. While he didn't pursue the notion of trigonometric series further, this work set the stage for the rest of his career. Most fundamentally, the nature of the results required him to consider arbitrary infinite collections of points on the real line - this directly motivated the development of set theory. Additionally, he conceived the notion of "derived sets" - what today is known as the set of limit points of a point set P. Although in these papers he only uses derived sets of the first order, later writings make clear he had already conceived of derived sets of the second order (the set of limit points of the first derived set) and beyond - into realms of infinities [3].

On a more ominous note, Kronecker had already begun to oppose Cantor's results. As a very fundamentalist mathematician, he wished to base all mathematics on the integers and finite arithmetic operations thereon - so the very notion of having arbitrary infinite point sets was counter to his program. Although he still encouraged Cantor at this stage, even helping to simplify some of his results, this was not to last for much longer [3].

2 The Birth of Set Theory

In 1874, Cantor published a paper entitled "On a Property of the System of all the Real Algebraic Numbers", marking the first appearance of set theory in mathematics [1]. Ostensibly, the paper sets out to prove that the set of algebraic numbers (i.e., numbers that are solutions to polynomial equations in one variable with integer coefficients) is countable - that is, can be listed in a fixed order, or to be put into a 1:1 correspondence with the integers (all three are equivalent formulations).

Proof. We must establish that a fixed ordering of the algebraic numbers exists. Define the height of an algebraic equation to be the sum $h = n + |a_0 + |a_1| + \cdots + |a_{n-1}| + |a_n|$, where n is the degree of the equation and a_i is the coefficient of x^{n-i}. We can then group all algebraic equations by height, starting at $n = 2$ ($n = a_0 = 1$; the equation $x = 0$) and continuing on in ascending order. Within each height-group, it is not difficult to establish a standard ordering (first order by ascending degree, then ascending initial coefficient, then ascending second coefficient, $\ldots$, then by ascending root magnitude, and finally by placing a negative root before a positive one). By discarding any roots previously obtained, we have now obtained a definite ordering of all algebraic numbers. QED. [1]

As an application of this result, Cantor goes on to corroborate Liouville's proof that there are an infinite number of transcendental (i.e., non-algebraic) numbers in any given interval. He does this in a rather roundabout and carefully worded way.

Proof. We wish to prove first that "given any sequence ω_n of real numbers and any interval $[\alpha, \beta]$, one can determine a number m in $[\alpha, \beta]$ that does not belong to ω_n. Hence, one can determine infinitely many numbers m in $[\alpha, \beta]$" [4].

We can assume that members of the sequence ω_n are distinct; only technical modifications are required to handle sequences with duplicate entries. Consider the first two elements of the sequence ω_n that fall within $[\alpha, \beta]$. From these, form the interval $[\alpha_1, \beta_1]$. Repeat the process indefinitely.

Case 1. The process terminates, resulting in a final interval $[\alpha_n, \beta_n]$. Then by definition of the process, there is at most one element of the sequence ω_n in this interval; choose m to be any other number in this interval apart from the endpoints.

Case 2. The process does not terminate. Let $\lim_{n\to\infty} \alpha_n = \alpha_\infty$; $\lim_{n\to\infty} \beta_n = \beta_\infty$. The limits are bounded above by b_1 and below by a_1, and each limit is monotonic by construction (a_n is monotonically increasing while b_n is monotonically decreasing). Since each sequence is monotonic and bounded, they each must have a limit.

Case 2a. $\alpha_\infty < \beta_\infty$. Then let m be any number in the interval $[\alpha_\infty, \beta_\infty]$.

Case 2b. $\alpha_\infty = \beta_\infty$. (NB: This holds true for the sequence of algebraic numbers.) Let $m = \alpha_\infty = \beta_\infty$. Suppose $m = \omega_i$ for some i. But ω_i is the endpoint for some constructed interval $[\alpha_j, \beta_j]$, and is thus not contained in $[\alpha_{j+1}, \beta_{j+1}]$, so cannot be the limit $\alpha_\infty = \beta_\infty$. Thus, $m = \omega_i$ is not true for any i, so m is not contained in the sequence. QED. [4]

As a corollary of this result, it simply happens to fall out that there are infinitely many transcendental reals in any interval of $\mathbb{R}$ (applying the results of the previous theorem to the series of algebraic numbers and any interval), and that the real numbers cannot be written in a sequence [4].

This last is the truly revolutionary part of Cantor's paper: the real numbers cannot be put into 1:1 correspondence with the natural numbers. Somehow, the infinity of the real numbers is actually larger than the infinity of the natural numbers. Why, then, didn't he just come out and say this? The answer comes back to Kronecker, one of the reviewers of *Crelle's Journal*, where Cantor published his paper. Had he come right out with these startling results on infinities, Kronecker would surely have blocked its publication; as it was, he simply corroborated a previously accepted result via means that simply happened to have revolutionary implications [3].

3 A Presentation of Point Sets

In the next half-decade, Cantor got married to Vally Guttman and published a paper on dimension (showing that a square contains the same number of points as a line). Kronecker delayed its publication in *Crelle's Journal*, so Cantor resolved never to publish in the journal again. Despite this, he was extremely productive in the next half-decade, publishing a series of six papers from 1879 to 1884 fleshing out some more ideas of set theory in the context of point sets [3].

The first four papers were relatively uneventful, mostly summarizing previous results and providing a few new applications. The first formally defined the *power* of a set as follows: "Two sets are said to be of the same power if a one-to-one correspondence between their respective elements is possible" [3]. In particular, he noted the difference between denumerable sets (equal in power to $\mathbb{N}$) and nondenumerable sets (equal in power to $\mathbb{R}$). The following three papers dealt with infinite derived sets of the second order, higher dimensions, and applications to functions, respectively [3].

The fifth paper was the most substantive of the six. Published in 1883, it was entitled *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, or *Foundations of a General Theory of Aggregates*. Here, Cantor presents for the first time in full his transfinite numbers, constructed via two principles of generation. The first is familiar as the means of constructing the natural numbers - the "successive addition of units" [3]. While $\mathbb{N}$ has no largest element, one can consider the order of the entire set, ω. This is the second principle of generation. Once we accept ω, we can then apply the first principle of generation to it, thereby constructing $\omega + 1, \omega + 2, \ldots$, all the way up to 2ω. In like manner, we can construct $3\omega, 4\omega, \ldots$ on up to ω^ω and beyond [5]. He proceeds to establish principles of transfinite arithmetic (which turns out to be non-commutative in general) [3].

While groundbreaking in their subject matter and scope, the fifth and sixth papers of Cantor's series left open an important question: Now that we know that the power of $\mathbb{N}$ is less than the power of $\mathbb{R}$, is there any set that has a power between the two? This question has come to be known as the "Continuum Hypothesis": Is there a set whose power is between that of the natural numbers and the reals? [3]

4 Disillusionment

As might be expected, Kronecker was less than satisfied with Cantor's presentation of transfinite, "actually infinite" numbers and began working to actively discredit his work. Cantor was also working feverishly to prove the continuum hypothesis with little success (he managed to "prove" it, "disprove" it, and then declare it an open question once more in 1884 alone). Likely as a combination of these factors, he suffered a nervous breakdown in 1884, unable to work for about a month. The following year, he submitted two articles to the *Acta Mathematica*, his journal of choice following his dissociation from *Crelle's*. However, the editor, Gösta Mittag-Leffler, declined to publish the papers, suggesting he ought instead to leave them buried, to be discovered a century hence when they might be appreciated for their worth [3].

It seems that this rejection caused Cantor to become disillusioned with mathematics for a time. He turned to Catholicism and began exploring a variety of other interests, including attempting to prove that Francis Bacon wrote Shakespeare's plays and freemasonry. He published several papers over the next few years, but in philosophical rather than mathematical journals. In 1891, he founded the *Deutsche Mathematiker-Vereinigung*, or German Mathematical Union. While this might seem to counter his recent disillusionment, it in fact furthered his dissociation from the traditional university-based mathematical beauracracy he had so come to despise [3]. At the inaugural meeting, he presented a new, more intuitive proof of the uncountability of $\mathbb{R}$ that is popularly known today as the "Diagionalization Argument":

Proof. Suppose you have an ordered list of the real numbers (i.e., you have put $\mathbb{R}$ into a 1:1 correspondence with $\mathbb{N}$). For simplicity's sake, consider a subset of this list, the set of real numbers in $(0, 1)$. Write each as an infinite decimal, obtaining a table of the following sort:

ω_1	**.1**	2	3	4	5	6	7	8	...
ω_2	.3	**3**	3	3	3	3	3	3	...
ω_3	.4	1	**2**	6	8	9	0	3	...
ω_4	.1	4	1	**5**	9	2	6	5	...
ω_5	.0	1	0	0	**1**	1	0	0	...

By considering the nth digit of ω_n, we can write a decimal $0.13251\ldots$. Then, by increasing each digit by 1 ($9 \to 0$), we can write the decimal $0.24362\ldots$. Since it differs from ω_n in the

nth place, this constructed number cannot be on the original list, contradicting the original claim that such a list exists. QED. [5]

5 Contributions to the Founding of the Theory of Transfinite Numbers

Following Kronecker's death in 1891, the intensely personal opposition to Cantor's work was gone. In 1895 and 1897, he published a pair of far-reaching articles entitled "Contributions to the Founding of the Theory of Transfinite Numbers", summarizing, justifying and furthering his work in this newfound realm. A comprehensive overview (and one of Cantor's only works translated into English), it is primarily via these articles that Cantor is known today [3].

The first article deals with simply-ordered sets: that is, sets for which each pair of elements possesses exactly one of the following relations: $a < b, a = b, a > b$. The work begins with a basic definition of a set (which, although modified in the years to come, nevertheless directly reflects our modern intuition of what a set is): "By an 'aggregate' we are to understand any collection into a whole M of definite and separable objects m of our intuition or our thought. These objects are called the 'elements' of M." [2] (Note that a set need not, by definition, contain numbers!) From here, Cantor performs a double abstraction (both of the nature of the elements and the order thereof) to construct the concept of cardinality (roughly, the size of a set). We can then begin considering 1:1 correspondences, as first presented back in 1874. Additionally, he introduces the notation $\aleph_0$ to represent the first transfinite cardinal number, i.e., denumerable infinity or the order of the set of all integers [2].

The second article deals with well-ordered sets: that is, sets like the integers which have a least element and in which each element has a unique successor. He had hoped to prove the continuum hypothesis by establishing that the cardinality of the continuum (known to be the power set of the real numbers, or $2^{\aleph_0}$) equals $\aleph_1$, known to be the second-smallest transfinite number. While he did establish and define $\aleph_1$ as the cardinality of the second number class (the ωs presented in Section 3) and was able to prove that it was the second-smallest transfinite number, he was unable to show that it equals the cardinality of the continuum [2, 3].

6 Reaching Into the Future

Following the publication of these two summary works, Cantor did not produce much more in the way of mathematics. He suffered another nervous breakdown in 1899, and his youngest son Rudolf died later that year. Hospitalized again in the winter term of the 1902-03 academic year, his breakdowns became more and more frequent until his death of heart failure on January 6, 1918. However, Cantor's work far outlived him, and his critical developments continue to be recognized to this day [3].

Even during his lifetime, his contributions gradually began to be accepted in the mathematical community. In 1900, the continuum hypothesis was selected as the first of the twenty-three Hilbert Problems [3]. He received the Sylvester Medal of the Royal Society of London in 1904, and the first textbook on set theory was published in 1906 [1].

By far the greatest problem Cantor left unsolved was that of the continuum hypothesis. As it turns out, there was good reason for this failure. In 1931, Gödel published his Incompleteness Theorem, proving that in any sufficiently robust axiomatic arithmetic system, there exist propositions that are undecidable within the system [5]. Nevertheless, Gödel went on to prove in 1936 that the continuum hypothesis was consistent with the other axioms of set theory. Finally, in 1963, Paul Cohen finally laid the matter to rest and proved that the continuum hypothesis is in fact an undecidable statement [3]. In a way, Kronecker's fears ended up being well-founded: the trail that Cantor blazed has no clear endpoint. Nevertheless, the mere act of following that trail created an entire new branch of mathematics and masterfully set the stage to make clear the indeterminacy lying at the heart of mathematics.

References

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